Math 254-2 Exam 8 Solutions

1. Carefully define the linear algebra term "linear mapping". Give two examples on $\mathbb{R}^2$.

   A linear mapping is a function f from a vector space to another vector space, that satisfies $f(u+v) = f(u) + f(v)$ and $f(cu) = cf(u)$, for all vectors u, v and all scalars c. Many examples are possible, such as $f(x,y) = (x,y), f(x,y) = (3x+y, 2x-4y), f(x,y) = (0,0), f(x,y) = (x,0)$.

2. Give any inner product on $\mathbb{R}^2$, OTHER than the dot product. Use your inner product to calculate $\langle u, v\rangle$ for $u = (1,3)^T, v = (2,-1)^T$.

   An inner product on $\mathbb{R}^n$ can be built from any positive definite matrix A via $\langle u, v\rangle = u^T Av$. 2×2 matrices are positive definite if both diagonal entries are positive and the determinant is positive. For example, we could take $A = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 3 \end{smallmatrix}\right)$, in which case the inner product is $\langle (x_1,y_1),(x_2,y_2)\rangle_A = 2x_1x_2+x_1y_2+x_2y_1+3y_1y_2$. Or, we could take $B = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right)$, in which case the inner product is $\langle (x_1,y_1),(x_2,y_2)\rangle_B = x_1x_2 + 2y_1y_2$. $\langle u, v\rangle$ depends on which inner product you specify; $\langle u, v\rangle_A = 4+6-1-9 = 0$, while $\langle u, v\rangle_B = 2-6 = -4$.

3. Find two different functions f, g on $\mathbb{R}$, with $f\circ f = g \circ g = 1_{\mathbb{R}}$.

   $f \circ f = 1_{\mathbb{R}}$ means that $f(f(x)) = 1_{\mathbb{R}}(x) = x$, for all $x \in \mathbb{R}$. There are two familiar functions that work: $f(x) = x,\ g(x) = -x$.

4. Consider all possible linear mappings from $\mathbb{R}^4$ to $\mathbb{R}^2$. What are the possible nullities and ranks of these? Give an example function for each possible combination, and indicate which functions are one-to-one and which are onto.

   The columnspace of a linear mapping is a subspace of the codomain ($\mathbb{R}^2$), hence is of dimension 0, 1, or 2. The domain ($\mathbb{R}^4$) has dimension 4, hence by the dimension theorem the nullity must be 4, 3, or 2 (respectively). Since the nullity cannot be zero, NONE of these functions can be one-to-one. To be onto, the rank must be 2. $f(x,y,z,w) = (x,w)$ has rank 2 (hence onto) and nullity 2. $g(x,y,z,w) = (x,x)$ has rank 1 and nullity 3. $h(x,y,z,w) = (0,0)$ has rank 0 and nullity 4. We can also express these linear mappings as matrix multiplications by $\left(\begin{smallmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{smallmatrix}\right)$, and $\left(\begin{smallmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{smallmatrix}\right)$, respectively.

5. Consider the mapping $F : \mathbb{R}_2[t] \to \mathbb{R}^2$ given by $F(p(t)) = (p(3), p(-1))$. Calculate $F(p(t))$ for $p(t) = t^2 - 3t + 1$. Determine whether F is linear.

   We have $F(t^2-3t+1) = (3^2-3\cdot 3+1, (-1)^2-3(-1)+1) = (1,5)$. This is a function from one vector space to another, hence for it to be linear it must satisfy closure. Let $p(t), q(t)$ be any polynomials in $\mathbb{R}_2[t]$; $F((p+q)(t)) = ((p+q)(3),(p+q)(-1)) = (p(3)+q(3), p(-1)+q(-1)) = (p(3),p(-1)) + (q(3),q(-1)) = F(p(t)) + F(q(t))$. Hence F is closed under vector addition. Let $p(t)$ be any polynomial in the domain, and c be any scalar. We have $F((cp)(t)) = ((cp)(3),(cp)(-1)) = (cp(3), cp(-1)) = c(p(3),p(-1)) = cF(p(t))$. Hence F is closed under scalar multiplication. This means that F *is* linear.